University of Mannheim
School of Social Sciences
Math Refresher for Political Science, Fall 2025
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Solutions Set Theory II

1. Let $S = \{\text{Homer, Marge, Bart, Lisa, Maggie}\}$. Enumerate the following relations.
   (a) "is a sibling of"
   $$R = \{(B, L), (B, Ma), (L, B), (L, Ma), (Ma, B), (Ma, L)\}$$
   (M) for Marge and (Ma) for Maggie.
   (b) "is married to"
   $$R = \{(H, M), (M, H)\}$$
   (c) "is taller than"
   $$R = \{(M, H), (M, B), (M, L), (M, Ma), (H, B), (H, L), (H, Ma), (B, L), (B, Ma), (L, Ma)\}$$
   (d) "is older than"
   $$R = \{(H, M), (H, B), (H, L), (H, Ma), (M, B), (M, L), (M, Ma), (B, L), (B, Ma), (L, Ma)\}$$

2. Let $S = \{1, 2, 3, 4\}$. Graph the following relations.
   (a) $=$

4				·
3			·	
2		·		
1	·			
=	1	2	3	4

   (b) $<$

4				
3				·
2			·	·
1		·	·	·
<	1	2	3	4

(c) R=$\{(1,1),(2,1),(2,2),(3,3),(4,3),(4,4)\}$

4			·	·
3			·	
2	·	·		
1	·			
R	1	2	3	4

3. Determine the following sets.

   (a) The upper contour set of Lisa in 1c.

   $$S = \{(B,L),(M,L),(H,L)\}$$

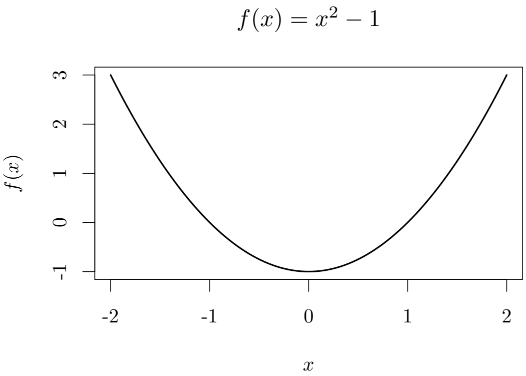
   (b) The lower contour set of Bart in 1d.

   $$S = \{(M,B),(H,B)\}$$

   (c) The upper contour set of 2 in 2b.

   $$S = \{(1,2)\}$$

   (d) Let $S = \mathbb{R}$. Determine the upper contour set of $x \in S$ given the relation $R = \{(x,y) \in S \times S | x^2 - 1 = y\}$. Graph the binary relation. Is the upper contour set convex or concave or none of both?



   The upper contour set is the area above the graph. It is convex.

4. Check whether the following relations are reflexive, irreflexive, transitive, complete, symmetric, antisymmetric and/or asymmetric. Also check whether they are a weak, strict, weak partial or strict partial order (or none of those).

   (a) $\leq$

   (b) $<$

   (c) $=$

   (d) $R_1 = \{(1,1),(1,2),(1,3),(1,4)\}$

(e) R_2 ="was born before"

(f) $R_3 = \{(a,a),(a,b)\}$

$\bullet$	ref	irr	trans	com	sym	anti	asym	weak	strict	partial
$\leq$	✓	$\bullet$	✓	✓	$\bullet$	✓	$\bullet$	✓		✓
$<$	$\bullet$	✓	✓	✓	$\bullet$	✓	✓		✓	✓
$=$	✓	$\bullet$	✓	✓	✓	✓	$\bullet$	✓		✓
R_1	✓	$\bullet$	✓	$\bullet$	$\bullet$	✓	✓			✓
R_2	$\bullet$	✓	✓	$\bullet$	$\bullet$	✓	$\bullet$			✓
R_3	✓	$\bullet$	✓	$\bullet$	$\bullet$	✓	$\bullet$			✓

5. Let $S = \{1,2,3\}$. Show by example that...

   (a) ...if R is asymmetric, it is also antisymmetric.
   An asymmetric relation, say $>$, is:

   $$R_1 = (2,1),(3,2),(3,1)$$

   For antisymmetry is must hold that $\forall x, y, \in S, [(xRy) \wedge (yRx)] \implies (x = y)$. Since the hypothesis is always false, the statement is always true and thus, the relation is antisymmetric.

   (b) ...if R is asymmetric, it is also irreflexive.
   Consider the same relation R_1. Irreflexivity requires that $\nexists x \in S, xRx$ which is the case.

   (c) ...if R is irreflexive and transitive, it is also asymmetric. Consider the irreflexive relation

   $$R_2 = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}.$$

   To meet also transitivity, the relation reduces to

   $$R_3 = \{(1,2),(1,3),(2,3)\}.$$

   It can be seen that this relation is also asymmetric.

   (d) ...if R is symmetric and antisymmetric, it is also transitive. As an example consider the relation $=$ which is symmetric and also antisymmetric.

   $$R_4 = \{(1,1),(2,2),(3,3)\}.$$

   This relation is also trnasitive since the hypothesis is always false and therefore the statement is true.

   As another example consider the relation $\neq$. This relation is symmetric, but it is not antisymmetric and therefore not necessarily transitive.

6. Let $A = \{\{a\},\{b\},\{a,b\}\}$. Let $B = \mathcal{P}(A) \setminus \emptyset$, where $\mathcal{P}(A)$ is the *power set*, the set of all subsets of A. Define a binary relation $R \equiv \subseteq$.

(a) Explicitly enumerate B, and state its cardinality.

The power set of A minus the empty set is

$$B = \{\{\{a\}\}, \{\{b\}\}, \{\{a,b\}\}, \{\{a\},\{b\}\}, \{\{a\},\{a,b\}\}, \{\{b\},\{a,b\}\}, \{\{a\},\{b\},\{a,b\}\}\}$$

Its cardinality is 7 ($2^3 - 1$).

(b) Prove whether R is a weak order on B.

A weak order requires reflexivity, completeness and transitivity. Consider the completeness requirement. $\{\{a\}\} \subseteq \{\{b\}\}$ is false. The reverse, $\{\{b\}\} \subseteq \{\{a\}\}$ is also false. Thus, the relation is not complete and therefore R is not a weak ordering on B.

(c) Prove whether R is a partial order on B.

A partial ordering requires reflexivity, transitivity and antisymmetry. Reflexivity requires that $\forall x$ in S, xRx. This is clearly the case, since $\subseteq$ includes equality and thus $\forall x$ in $B, x = x$. Transitivity is also given: If $x \subseteq y$ and $y \subseteq z$, then $x \subseteq z$. For antisymmetry, it needs to hold that (xRy) and (yRx) imply $x = y$. The only case for which the premise can hold for the relation $\subseteq$ is if $x = y$. Thus, $\subseteq$ is a partial order on B.

(a) Let R_1 and R_2 be transitive relations on a set S. Does it follow that $R_1 \cup R_2$ is transitive?

No. A counterexample: $S = \{1,2\}$. $R_1 =' >' = \{(2,1)\}$ and $R_2 =' <' = \{1,2\}$. And $R_1 \cup R_2 = \{(1,2),(2,1)\}$ is not transitive.

(b) Let R_1 and R_2 be transitive relations on a set S. Does it follow that $R_1 \cap R_2$ is transitive?

Yes. Assume R_1 and R_2 are both transitive and let $(a,b),(b,c) \in R_1 \cap R_2$. Then $(a,b),(b,c) \in R_1$ and $(a,b),(b,c) \in R_2$. It is given that both R and S are transitive, so $(a,c) \in R_1$ and $(a,c) \in R_2$. Therefore $(a,c) \in R_1 \cap R_2$. This shows that for arbitrary $(a,b),(b,c) \in R_1 \cap R_2$ we have $(a,c) \in R \cap R_2$. Thus $R_1 \cap R_2$ is transitive.